

**LIGHT DYNAMICS IN NONLINEAR TRIMERS AND TWISTED
MULTICORE FIBERS**

CLAUDIA CASTRO-CASTRO*

*Department of Mathematics, Southern Methodist University, Dallas, Texas 75275, USA
ccastrocastr@smu.edu*

YANNAN SHEN

*Department of Mathematics, University of California Northridge, California 91330, USA
yannan.shen@csun.edu*

GOWRI SRINIVASAN

Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87544

ALEJANDRO B. ACEVES

Department of Mathematics, Southern Methodist University, Dallas, Texas 75275, USA

PANAYOTIS G. KEVREKIDIS

*Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA
01003-4515, USA*

Received (Day Month Year)

Novel photonic structures such as multi-core fibers and graphene based arrays present unique opportunities to manipulate and control the propagation of light. Here we discuss nonlinear dynamics for structures with a few (2 to 6) elements for which linear and nonlinear properties can be tuned. Specifically we show how nonlinearity, coupling, and parity-time (\mathcal{PT}) symmetric gain/loss relate to existence, stability and in general, dynamical properties of nonlinear optical modes. The main emphasis of our presentation will be on systems with few degrees of freedom, most notably couplers, trimers and generalizations thereof to systems with 6 nodes.

Keywords: Fiber optics; \mathcal{PT} -symmetry; discrete NLS

Introduction

Advancements in photonics are driven by the possibilities they present to all optical devices. Complex array geometries and the use of novel materials such as graphene and metamaterials for which one can tune properties such as coupling, nonlinearity and gain/loss translate into controlled dynamics and bifurcations of interest to logical devices (switches, transistors, logic gates), next generation lasers and new formats for optical communications.¹

*Corresponding author.

Here, we follow the vein of the work of Bender *et al.*² who considered whether a quantum mechanical Hamiltonian system with a complex potential can have a real spectrum. It is the case that a necessary condition for the spectrum to be real is that the Hamiltonian of the Schrödinger has parity \mathcal{P} and time \mathcal{T} symmetry (\mathcal{PT}) which is true when the potential satisfies the condition $V(x) = V^*(-x)$.³ This has led to the identification of a broad range of special classes of \mathcal{PT} -symmetric Hamiltonians which possess a real eigenspectrum and \mathcal{PT} -symmetric eigenstates. Perhaps more relevant is that if the strength of the odd imaginary part of the potential is parametrized, one typically finds that at a certain parameter value the spectrum becomes complex. Whenever the eigenfrequencies remain real, we refer to this case as *unbroken* \mathcal{PT} -symmetry. If they become complex, one typically refers to this as the regime where \mathcal{PT} -symmetry has been *broken*. This work originally presented in the quantum mechanical framework can be naturally extended to the optical regime. In fact, it was in the latter setting that \mathcal{PT} -symmetric systems were first experimentally realized in the form of dimers,^{4,5} (i.e., optical waveguide couplers) and more recently in the realm of full one-dimensional arrays.⁶ Novel photonic structures can be conceived with \mathcal{PT} -symmetry properties, which translate to having a real index of refraction profile that is even in space, whereas the gain/loss imaginary part is odd. Manipulating light propagation under \mathcal{PT} -symmetric structures and what broken symmetry entails has been the focus of intense research activity in recent time.^{7,8}

In this paper we present results on nonlinear \mathcal{PT} -symmetric discrete arrays focusing especially on the case of a few lattice nodes. In such photonic systems there are four principal parameters that characterize the type of nonlinear modes, their propagation features and possible bifurcations: nearest neighbor coupling, individual propagation constants, nonlinear coefficient and \mathcal{PT} gain-loss coefficients. Altogether, dynamics in nonlinear photonic arrays is currently a rich field of active research. Here we present results for three configurations: a dimer, a trimer and a most recently considered setting of twisted multicore fibers.⁹

The paper is organized as follows. In Section 1, we introduce the coupled \mathcal{PT} -symmetric nonlinear Schrödinger (NLS) array with two and three sites. We will refer to these cases as coupler and trimer respectively. While couplers and trimers have been thoroughly studied, this section primarily serves to illustrate \mathcal{PT} dynamics in the simplest case. Section 2 presents our recent work which extends that of Ref. 9 by inclusion of nonlinear effects. As a way of comparison, we present numerical results for a conservative set up, and of cases with gain or loss in the individual cores. Section 3 summarizes our results and presents some possibilities for further study.

1. \mathcal{PT} -symmetric coupler and trimer

Consider the propagation dynamics in a discrete array of N optical fibers in Kerr media described by the one-dimensional discrete Nonlinear Schrödinger Equation

(DNLSE):

$$i\dot{\psi}_n = \epsilon_n \psi_n + \sigma |\psi_n|^2 \psi_n - c(\psi_{n+1} + \psi_{n-1}), \quad (1.1)$$

where n is the waveguide number, the dot denotes derivative with respect to the direction of propagation z , σ is the strength of the Kerr-nonlinearity, c is the uniform strength of the nearest neighbor coupling, ϵ_n represents the on-site refractive-index profile (Fig. 1). The Hamiltonian that gives rise to the equations of motion is given by the equation

$$H_D = \sum_n \epsilon_n |\psi_n|^2 + \frac{\sigma}{2} |\psi_n|^4 - c (\psi_{n+1} \psi_n^* + \psi_{n+1}^* \psi_n). \quad (1.2)$$

We start considering an ideal configuration of identical optical wave guides, where ϵ_n is a fixed constant ϵ .

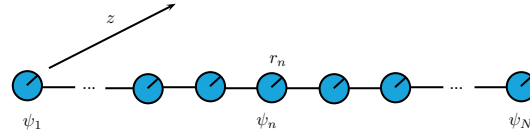


Fig. 1. Diagrammatic representation of a planar optical waveguide array. r_n is the radius of each particular fiber.

1.1. The coupler

We explore the eigenspectrum associated with a structure which has complex valued Hamiltonian with \mathcal{PT} -symmetry composed of two waveguides, one with loss at location $n = 1$ and the other with gain at the adjacent waveguide $n = 2$. That is $\epsilon_n = \alpha_n + i\gamma\beta_n$, where γ defines the rate of gain/loss at each waveguide, and α_n and β_n satisfy the even and odd conditions $\alpha_2 = \alpha_1$ and $\beta_2 = -\beta_1$. The propagation equations for this case can be written as

$$i\dot{\psi}_1 = \delta\sigma |\psi_1|^2 \psi_1 - c\psi_2 + \alpha\psi_1 + i\gamma\beta\psi_1, \quad (1.3a)$$

$$i\dot{\psi}_2 = \delta\sigma |\psi_2|^2 \psi_2 - c\psi_1 + \alpha\psi_2 - i\gamma\beta\psi_2. \quad (1.3b)$$

It is well known based on the work of Ramezani *et al.*¹⁰ that the coupler is an integrable dynamical system due to the existence of two conserved quantities in the form of $C^2 = 4 [\delta^2 \sigma^2 |\psi_1|^2 |\psi_2|^2 - \delta\sigma (\psi_1^* \psi_2 + \psi_1 \psi_2^*) + 1]$, and $J = |\psi_1|^2 + |\psi_2|^2 + (2\beta/\delta\sigma) \sin^{-1} \{[\delta\sigma (\psi_1^* \psi_2 + \psi_1 \psi_2^*) - 2]/C\}$. Moreover, the integrability has been illustrated in a complementary fashion by Barashenkov¹¹ who showed that upon suitable coordinate transformations, the system can be turned into a (conservative) Hamiltonian one.

In the weakly nonlinear regime, we have developed a modulation theory for the coupler problem. Observe that the linear model has eigenvalues $\lambda_{1,2} = \alpha \pm$

$\sqrt{c^2 - \beta^2 \gamma^2}$ which are real as long as the gain/loss parameter γ is smaller than a critical value $\gamma_c = c/\beta$. The respective eigenvectors are

$$\begin{pmatrix} -e^{i\theta} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix},$$

where θ is defined such that $\sin \theta = \beta\gamma/c$.

In the case of weak nonlinearity, the evolution equations read

$$i\dot{\psi} = VDV^{-1}\psi + \delta\sigma N(\psi), \quad (1.4)$$

where $\psi = (\psi_1, \psi_2)^T$, VDV^{-1} is the diagonal decomposition of the block matrix associated with the linear eigenvalue problem, δ is a small parameter, and the nonlinear term is given by

$$N(\psi) = \begin{bmatrix} |\psi_1|^2 \psi_1 \\ |\psi_2|^2 \psi_2 \end{bmatrix} = \begin{bmatrix} |\psi_1|^2 & 0 \\ 0 & |\psi_2|^2 \end{bmatrix} \psi.$$

Multiplying by V^{-1} from the left on both sides of (1.4) and defining $\psi = V\phi$, we can rewrite Eq. (1.4) as

$$i\dot{\phi} = D\phi + \delta\sigma V^{-1}N(V\phi). \quad (1.5)$$

From the linear solution of the previous equation in terms of ϕ , we can write the solution to the initial value problem as $\phi(z) = e^{-iDz}C$, where $C = C(z)$

$$i\frac{dC}{dz} = \delta\sigma e^{iDz}V^{-1} \begin{bmatrix} |\psi_1|^2 & 0 \\ 0 & |\psi_2|^2 \end{bmatrix} V e^{-iDz}C \quad (1.6)$$

For brevity, we restrict the model to two scenarios: *a*) we assume \mathcal{PT} -symmetry as small perturbation of the original system, $\alpha = 1$, $\beta = \delta$, where δ is a small parameter, $\delta \ll 1$. In the second case *b*) we consider the regime near the \mathcal{PT} -symmetry transition region; that is when γ is similar to c . We aim to derive amplitude equations as slowly varying functions of time. In the first case *a*), the right hand side of (1.6) has oscillating terms with frequencies proportional to $\lambda_1 - \lambda_2$, which can also be written as $2c\cos\theta$, where $\sin\theta = \delta\gamma/c$. These terms represent a fast oscillating evolution when δ is a small parameter. We thus perform an averaging and keep only the slow oscillating (near resonant) terms. Introducing the scale variable $Z = \delta z$, we can represent the amplitudes as slowly varying functions of Z : $C_k(z; \delta) = \tilde{C}_k(Z)$ whose evolution is given by

$$i\frac{d\tilde{C}_1}{dZ} = \sigma \left(|\tilde{C}_1|^2 \tilde{C}_1 + 2|\tilde{C}_2|^2 \tilde{C}_1 \right), \quad (1.7a)$$

$$i\frac{d\tilde{C}_2}{dZ} = \sigma \left(|\tilde{C}_2|^2 \tilde{C}_2 + 2|\tilde{C}_1|^2 \tilde{C}_2 \right). \quad (1.7b)$$

We note that system (1.7) can be solved analytically given the existence of two conserved quantities, $|\tilde{C}_1|^2$ and $|\tilde{C}_2|^2$. The resulting dynamics is a trivial phase modulation to the linear modes.

A more interesting case is b), representing the regime near the transition region. We can no longer average the terms in (1.6) with frequency proportional to $\lambda_1 - \lambda_2 = 2c \cos \tilde{\theta}$, where $\sin \tilde{\theta} = \gamma/c$, which is close to zero. Let $k = (\lambda_1 - \lambda_2)/2\delta$, with $k = \mathcal{O}(1)$. Introducing again the slow scale variable $Z = \delta z$ and using the definition of the parameter k , the system of amplitude equations can be written as

$$i \frac{d\tilde{C}_1}{dZ} = \sigma \left(|\tilde{C}_1|^2 \tilde{C}_1 + 2|\tilde{C}_2|^2 \tilde{C}_1 + 2e^{-2ikZ} \tilde{C}_1^2 \tilde{C}_2^* \right) - \gamma e^{4ikZ} \tilde{C}_1^* \tilde{C}_2^2, \quad (1.8a)$$

$$i \frac{d\tilde{C}_2}{dZ} = \sigma \left(|\tilde{C}_2|^2 \tilde{C}_2 + 2|\tilde{C}_1|^2 \tilde{C}_2 + 2e^{2ikZ} \tilde{C}_1^* \tilde{C}_2^2 \right) - \gamma e^{-4ikZ} \tilde{C}_1^2 \tilde{C}_2^*. \quad (1.8b)$$

One finds that the squared amplitudes satisfy:

$$\frac{d|\tilde{C}_1|^2}{dZ} = 2\sigma \left[2|\tilde{C}_1|^2 \operatorname{Im} \left(e^{-2ikZ} \tilde{C}_1 \tilde{C}_2^* \right) + \operatorname{Im} \left(e^{-4ikZ} \tilde{C}_1^2 \tilde{C}_2^{*2} \right) \right], \quad (1.9a)$$

$$\frac{d|\tilde{C}_2|^2}{dZ} = 2\sigma \left[2|\tilde{C}_2|^2 \operatorname{Im} \left(e^{2ikZ} \tilde{C}_1^* \tilde{C}_2 \right) + \operatorname{Im} \left(e^{4ikZ} \tilde{C}_1^{*2} \tilde{C}_2^2 \right) \right], \quad (1.9b)$$

which in particular shows that the system is no longer integrable, but the amplitudes do not live in a circle. Future work will expand on the full analysis of the weakly nonlinear regime, in particular the case near transition.

1.2. The trimer

For the trimer, here we only address the question of whether a region of unbroken \mathcal{PT} -symmetry persists in the linear case for on-site refractive-index profile,

$$\epsilon_n = \alpha_n + i\delta\gamma\beta_n,$$

where δ is a small parameter, γ defines the rate of gain/loss at waveguides, and α_n and β_n satisfy the even and odd conditions $\alpha_n = \alpha_{-n}$ and $\beta_n = -\beta_{-n}$. For an open linear trimer with an active, a lossy, and a conservative waveguide, we seek solutions of the form $\psi_n = A_n e^{-i\lambda z}$, then (A_1, A_2, A_3) satisfies

$$\lambda A_1 = \epsilon_1 A_1 - c A_2$$

$$\lambda A_2 = \epsilon_2 A_2 - c(A_3 + A_1)$$

$$\lambda A_3 = \epsilon_3 A_3 - c A_2$$

For the \mathcal{PT} -symmetric case of a homogeneous array with $\alpha_n = \beta_n = 1$ the characteristic polynomial of the associated matrix problem is

$$p(c, \gamma) = -2c^2 \lambda + 2c^2 + \lambda^3 - 2\lambda^2 + \lambda + \lambda\gamma^2.$$

In order to explore whether a region of transition (real to complex eigenvalues) exists, we check the sign of the discriminant of the third degree polynomial associated with the nontrivial roots of $p(c, \gamma)$ is

$$\begin{aligned} \Delta = & -108c^4 - 72c^2(-2c^2 + \gamma^2 + 1) - 4(-2c^2 + \gamma^2 + 1)^3 \\ & + 4(-2c^2 + \gamma^2 + 1)^2 + 64c^2 \end{aligned} \quad (1.10)$$

In Figure 2 we plot the discriminant as a function of c and γ as a red surface and the plane $\Delta = 0$ as a blue surface. We observe that there exists a region of the (c, γ) plane where real eigenfrequencies exist whenever the discriminant is positive. We also note a transition region from broken to unbroken \mathcal{PT} -symmetry, or viceversa, is represented as the intersection curve of the red and blue surfaces.

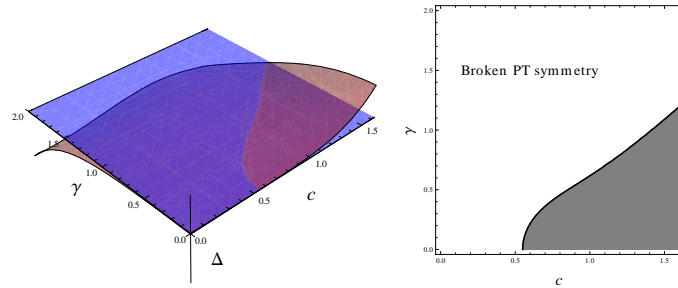


Fig. 2. (Color online) Discriminant as function of coupling c and gain/loss rate γ . The shaded region in the right panel shows the region of unbroken \mathcal{PT} -symmetry in the (c, γ) plane.

Future work will explore the dynamics for parameters in the vicinity of $\Delta(\alpha, \gamma, c) = 0$ which will include weak nonlinear effects.

2. Twisted multicore fibers

In this section, we consider the beam propagation dynamics in a discrete multicore fiber of N optical waveguides arranged equally spaced in a ring of radius R_0 where each core has radius r_0 (Figure 3), described by the coupled-mode equations (CMEs)

$$i \frac{dc_n}{dz} = k (e^{-i\phi} c_{n+1} + e^{i\phi} c_{n-1}) + i\gamma_n c_n + \delta d |c_n|^2 c_n. \quad (2.1)$$

Here $n = 1, 2, \dots, N$ is the waveguide (site) number, c_n represents complex-valued amplitudes that depend on the direction of propagation variable $z \in \mathbb{R}$, k is the uniform strength of the nearest neighbor coupling, γ_n is the optical gain or loss rate at site n . The fiber is considered to be twisted along the propagation direction z with a twist rate $\varepsilon = 2\pi/\Lambda$, Λ is the spatial period. Recent work by Longhi⁹ only considers the linear $\delta d = 0$ case where he shows that transition from unbroken to broken \mathcal{PT} phases can be conveniently controlled by a suitable geometric twist of the fiber. The fiber twist which is responsible for the introduction of additional phase terms in the propagating fibre modes $e^{\pm i\phi}$, ϕ is defined as the Peierls phase,¹² $\phi = 4\pi^2 \varepsilon n_s r_0^2 / N\lambda$, where n_s is the substrate (cladding) refractive index, $r_0 \simeq R_0$ is an effective ring radius, λ is the wavelength of the propagating field. As such, in Ref. 12 the author finds that for three fibers with gain loss $\gamma_1 = 0, \gamma_2 = -\gamma_3 = \gamma$, the unbroken parity time phase is obtained when the condition $(\gamma/k)^2 <$

$3\sqrt{1 - \cos^{2/3}(3\phi)}$ is satisfied. We want to extend this work to larger arrays and account for nonlinear effects.

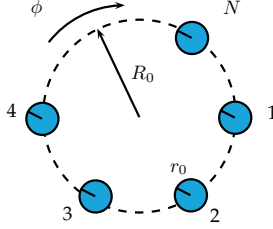


Fig. 3. Diagrammatic representation of a multi-core fiber. R_0 radius of the circular fiber and r_0 is the radius of each particular fiber. The fiber is twisted along the propagation direction z .

From now on, we will refer to equation (2.1) as twisted \mathcal{PT} -DNLS equation, which can be rewritten as

$$i\dot{\mathbf{c}} = (k\mathcal{K} + iG)\mathbf{c} + \delta dN(\mathbf{c})\mathbf{c}, \quad (2.2)$$

where $\mathbf{c} = (c_1, c_2, \dots, c_N)^T$, $G = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_N)$, and $N(\mathbf{c}) = \text{diag}(|c_1|^2, |c_2|^2, \dots, |c_N|^2)$ which describe the dissipation and the nonlinear part of the system, and \mathcal{K} is the coupling matrix

$$\mathcal{K} = \begin{pmatrix} 0 & e^{-i\phi} & 0 & & e^{i\phi} \\ e^{i\phi} & 0 & e^{-i\phi} & & \\ 0 & e^{i\phi} & 0 & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & 0 & e^{-i\phi} \\ e^{-i\phi} & & & & e^{i\phi} & 0 \end{pmatrix}_{N \times N}.$$

At this time we will only present our recent result on the existence of localized modes in the conservative case. For this, consider the conservative variant of the twisted fiber model, where $\gamma_n = 0$. In that case, the equations read:

$$i \frac{dc_n}{dz} = k(e^{-i\phi}c_{n+1} + e^{i\phi}c_{n-1}) + \delta d|c_n|^2c_n,$$

with periodic boundary conditions $c_{n+N} = c_n$. If we make the transformation $c_n = a_n(z)e^{i\varphi_n}$, $\varphi_{n+1} - \varphi_n = \phi$, the equations for a_n are those of the classical uniform array, but now the periodic boundary condition is modified to $a_{n+N} = a_ne^{iN\phi}$. This transformation also shows in the Hamiltonian

$$H = \sum_{n=1}^N k(c_{n+1}c_n^*e^{-i\phi} + c_nc_{n+1}^*e^{i\phi}) + \frac{\delta d}{2}|c_n|^4 = \sum_{n=1}^N k(a_{n+1}a_n^* + a_na_{n+1}^*) + \frac{\delta d}{2}|a_n|^4.$$

In the conservative case, the total power $P = \sum_{n=1}^N |c_n|^2$ is also an invariant. Interestingly, in the general case the evolution of P does not depend explicitly on the phase ϕ ,

$$\frac{d}{dz} \sum_{n=1}^N |c_n|^2 = \sum_{n=1}^N 2\gamma_n |c_n|^2.$$

One can see that for small γ_n the power is slowly varying and our numerics suggests an oscillatory pattern (see figure 5). While we have not determined a critical value of gain/loss for symmetry breaking dynamics, one can speculate that as in the case of the coupler, if at some point all energies propagate in the active (gain) fibers, then $\frac{dP}{dz} > 0$. Where the phase ϕ plays a role not explicitly seen in the dynamics of P is how this transition to dominant flow in the fibers with gain is determined by the twist. We are currently exploring such transition.

As it is the case with the untwisted multi-core system with no loss/gain, localized modes extend to the case where there is twist. One can find them by use of asymptotics, so that to leading order, by assuming that the peak intensity is in fiber $n = 1$ and in the case $N = 6$

$$\begin{aligned} i \frac{dc_1^{(0)}}{dz} &= \delta d |c_1^{(0)}|^2 c_1^{(0)}, \\ i \frac{dc_2^{(0)}}{dz} &= k e^{i\phi} c_1^{(0)}, \\ i \frac{dc_6^{(0)}}{dz} &= k e^{-i\phi} c_1^{(0)}, \\ i \frac{dc_3^{(0)}}{dz} &= k e^{i\phi} c_2^{(0)}, \\ i \frac{dc_5^{(0)}}{dz} &= k e^{-i\phi} c_6^{(0)}, \\ i \frac{dc_4^{(0)}}{dz} &= k e^{i\phi} c_3^{(0)} + k e^{-i\phi} c_5^{(0)}, \end{aligned}$$

with solutions

$$\begin{aligned} c_1^{(0)} &= \rho e^{-i\delta d \rho^2 z}, \\ c_2^{(0)} &= \frac{k}{\delta d \rho} e^{-i\delta d \rho^2 z + i\phi}, \\ c_6^{(0)} &= \frac{k}{\delta d \rho} e^{-i\epsilon_s d \rho^2 z - i\phi}, \\ c_3^{(0)} &= \frac{k^2}{(\delta d)^2 \rho^3} e^{-i\delta d \rho^2 z + 2i\phi}, \\ c_5^{(0)} &= \frac{k^2}{(\delta d)^2 \rho^3} e^{-i\delta d \rho^2 z - 2i\phi}, \\ c_4^{(0)} &= \frac{2k^3}{(\delta d)^3 \rho^5} \cos 3\phi e^{-i\delta d \rho^2 z}, \end{aligned}$$

where $\rho \gg 1$. Observe the particular amplitude dependence of $c_4(0)$ on the twist induced phase ϕ . For example, if $\phi = (2m + 1)\pi/6$, then $c_4(0) = 0$.

We now present numerical results obtained by integration of Eq. (2.1). First we consider a conservative case with $\gamma_n = 0$ for the defocusing case, $\delta d = -1$, and parameters $n_s = 1.552$, $R_0 = 8\mu m$, $\lambda = 980nm$, $r_0 = 0.91R_0$, $k = 3.59cm^{-1}$, and $\gamma = 2.05$. The fiber is initially excited in core 1. Results in Fig. 4 present the amplitudes of cores 1 and 4 which clearly highlights the role of the twist in a dramatic fashion. On the other hand, this behavior is reminiscent of the linear model treated in Ref. 9 suggesting that the topological feature of the twist and its effect on the dynamics is quite robust to nonlinear contributions which mainly affect the individual phases.

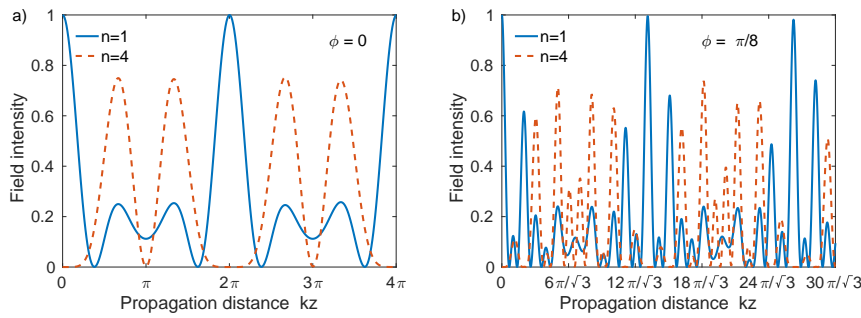


Fig. 4. (Color online) Evolution of field intensities for cores 1 (solid) and 4 (dashed) in the nonlinear, conservative, and defocusing case with $\gamma = 0$. Left and right panels correspond to non-twisted and twisted scenarios respectively.

Considering instead the nonlinear CMEs for the six-core fiber with alternating gain and loss $\gamma_n = (-1)^{n+1}\gamma$, we present numerical results of the integration of Eq. (2.1) in Fig. 5 for a defocusing case. The figure contains un(twisted) cases in top(bottom) panels. Values used were $n_s = 1.552$, $R_0 = 8\mu m$, $\lambda = 980nm$, $r_0 = 0.91R_0$, $\gamma = 0.1$, $k = 3.59cm^{-1}$, and $\delta d = -1$. As in the conservative case, only core 1 is initially excited. We observe there is an oscillatory behavior of the total power in the absence of twist, oscillating within a small window of the expected value of the conserved case. Similarly for the twisted case $\phi = \pi/8$.

In summary, the cases considered are examples of how the induced geometrical twist along with nonlinearity can cause power weakening/intensification in \mathcal{PT} -symmetric fibers. It would be of interest to extend this work for a large array and for concentric multi-core arrays.

3. Discussion

We have discussed some case examples of \mathcal{PT} -symmetric systems in the context of examples stemming from optical waveguide arrays. For a weakly nonlinear \mathcal{PT} -

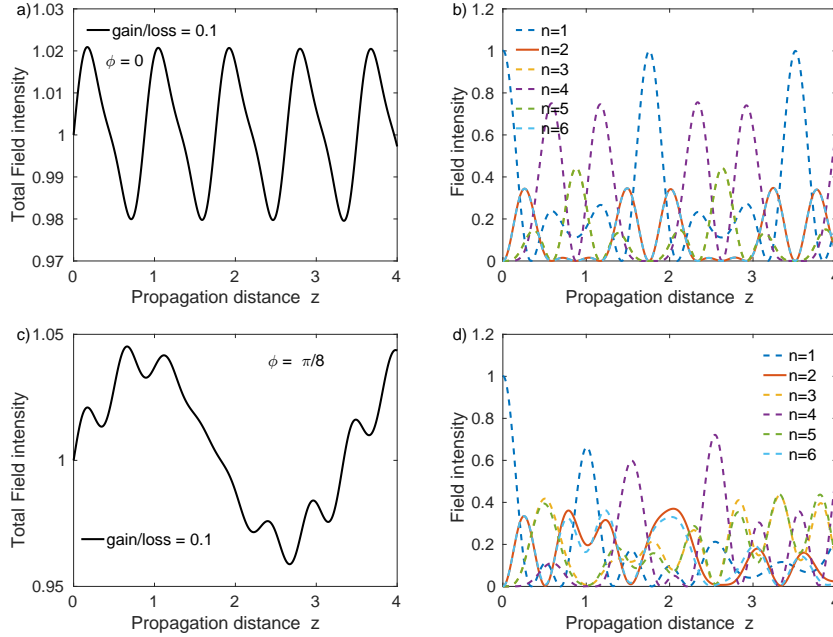


Fig. 5. (Color online) Evolution of total power (left), and individual power for cores 1-6 (right) in the nonlinear case with alternating gain-loss in the defocusing regime $\delta d = -1$ with $\gamma = 0.1$. Untwisted and twisted scenarios depicted on top and bottom panels respectively.

symmetric dimer, we have found the governing equations of the slow varying amplitudes of the mode solutions. We present two conserved quantities for the nonlinear \mathcal{PT} -symmetric dimer that we used to diagnose the performance of the numerical integrator. For a particular layout of the optical lattice with three waveguides, we show that there exists a parametric transition region from unbroken to broken \mathcal{PT} -symmetry. Of great interest will be to extend these notions to disordered arrays of waveguides. In the second part of this work we explore how the dynamics of nonlinear fiber with six cores is affected by an induced twist in the scenario when gain/loss is not present in the model and when there is alternating gain/loss profile. The numerical results highlight the potential of inducing a fiber twist to control the light dynamics in nonlinear multi-core fibers and suggest a rich scenario for further exploration of parameter space and extend this work to for large arrays and arrays with twist (z -dependent) management. Thus, extending present considerations towards identifying localized modes and their dynamical robustness both in the realm of one dimensional arrays, as well as in higher dimensional settings is one of the natural directions for future study.

Acknowledgments

Castro-Castro acknowledges partial support from LANL grant number LA-UR-15-26399.

References

1. S. Suchkov, A. A. Sukhorukov, J. Huang, S. V. Dmitriev, C. Lee, and Y. S. Kivshar, Nonlinear switching and solitons in \mathcal{PT} symmetric photonic systems, *Laser & Photonics Reviews* **10**(2) (2016) 177213.
2. C. M. Bender, Making sense of non-Hermitian Hamiltonians, *Reports on Progress in Physics* **70**(6) (2007) 947-1018.
3. A. E. Miroshnichenko, B. A. Malomed, and Y. S. Kivshar, Nonlinearly \mathcal{PT} -symmetric systems: spontaneous symmetry breaking and transmission resonances, *Physical Review A* **84**(1) (2011) 012123.
4. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Observation of parity-time symmetry in optics, *Nature Physics* **6**(3) (2010) 192-195.
5. A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Observation of \mathcal{PT} -symmetry breaking in complex optical potentials, *Physical Review Letters* **103**(9) (2009) 093902.
6. M. Wimmer, A. Regensburger, M.-A. Miri, C. Bersch, D. N. Christodoulides, and U. Peschel, Observation of optical solitons in \mathcal{PT} -symmetric lattices, *Nature Communications* **6** (2015) 7782.
7. S. Nixon, and J. Yang, Nonlinear wave dynamics near phase transition in \mathcal{PT} -symmetric localized potentials, *Physica D: Nonlinear Phenomena* **331** (2016) 48-57.
8. A. Sukhorukov, S. V. Dmitriev, S. V. Suchkov, and Y. S. Kivshar, Nonlocality in \mathcal{PT} -symmetric waveguide arrays with gain and loss, *Optics Letters* **37**(11) (2012) 2148-2150.
9. S. Longhi, \mathcal{PT} -phase control in circular multi-core fibers, *Optics Letters* **41**(9) (2016) 1897-1900.
10. H. Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides, Unidirectional nonlinear \mathcal{PT} -symmetric optical structures, *Physical Review A* **82**(4) (2010) 043803.
11. I.V. Barashenkov, 2014. Hamiltonian formulation of the standard \mathcal{PT} -symmetric nonlinear Schrödinger dimer. *Physical Review A* **90**(4) 045802.
12. S. Longhi, Light transfer control and diffraction management in circular fibre waveguide arrays, *Journal of Physics B: Atomic, Molecular and Optical Physics* **40**(23) (2007) 4477.